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Bifurcation Analysis for a Class of Cubic Switching Systems*

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In this paper, we consider a class of switching systems perturbed by cubic homogeneous polynomials. This class of systems is separated by a straight line: $y = 0$, and has three equilibria: $(0, 0)$ and $(\pm 1, 0)$ which are in the separation line. A new version of the Gasull–Torregrosa method based on Poincaré return maps is presented, and used to compute the Lyapunov constants. Based on this method, a complete classification on the center conditions is obtained for the studied class of systems. Furthermore, by perturbing the cubic switching integral system with cubic homogeneous polynomials, we show that at least ten small-amplitude limit cycles are obtained around one of the centres. This is a new lower bound for the number of limit cycles bifurcating from a center in such switching systems with cubic homogeneous nonlinearities.

Keywords: Switching system; Lyapunov constant; center; limit cycle; Poincaré return map.

1. Introduction

Determining the number and configuration of limit cycles in differential dynamical systems is related to the second part of Hilbert's 16th problem. An enormous amount of work has been done on this problem for continuous differential dynamical systems, see, e.g. [Ll'yashenko & Yakovenko, 1991; Shi, 1980; Wang, 1991, 1990; Yu & Han, 2012; Li & Liu, 2010; Li *et al.*, 2009]. In mechanics, electrical engineering and automatic control many problems are described by differential dynamical systems which are not continuous (nonsmooth). In the past few decades, there has been increasingly high interest in the qualitative analysis of discontinuous dynamical systems [Andronov *et al.*, 1959; Banerjee & Verghese, 2001]. In such systems, there exist not only the classical bifurcations, but also bifurcations of other types like the border-collision bifurcation

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[Zou *et al.*, 2006; Simpson & Meiss, 2007]. Switching systems studied in this paper are special kind of discontinuous differential dynamical systems, which are widely used for mathematical modeling in control and engineering.

Many researchers have investigated the switching systems. On one hand, some authors focused on the methods of bifurcation analysis in switching systems. Filippov [1988] established basic qualitative theory for switching systems. Coll and Gasull [Coll *et al.*, 2001] derived formulas for computing the first three Lyapunov quantities associated with three types of singularities. Kukučka [2007] showed the existence of a homoclinic solution in a perturbed nonsmooth system. Li and Huang [2014] considered the concurrent homoclinic bifurcation and Hopf bifurcation for a class of planar perturbed nonsmooth Filippov systems. Gasull and Torregrosa [2003] developed a method for constructing five limit cycles in a quadratic switching system, while only four limit cycles have been constructed for planar quadratic continuous differential systems [Shi, 1980; Sun & Shu, 1979].

On the other hand, more authors investigated the center and limit cycle problems in switching systems [Lunkevich, 1968; Filippov, 1988; Pleshkan & Sibirskii, 1973]. Center conditions have been established for the switching Kukles system [Gasull & Torregrosa, 2003] and the switching Liénard system [Coll *et al.*, 1999]. Han and Zhang [2010] proved that two limit cycles can bifurcate from a focus for piecewise linear systems. Chen and Du [2010] constructed a switching Bautin system and proved that nine limit cycles can bifurcate from a center of the system. Tian and Yu [2015] provided a complete classification on the conditions of a singular point being a center in the switching Bautin system, and constructed an example to show the existence of ten limit cycles bifurcating from the center. Recently, a planar quadratic switching system (the switching line that is not straight) has been constructed to obtain 16 limit cycles [Cruz *et al.*, 2019] by using the averaging approach up to ε^2 order.

However, very few works focus on the center and limit cycle problems in cubic switching systems, let alone the switching systems with homogeneous nonlinearities. Guo *et al.* [2019] studied a class of Z_2-equivariant cubic switching systems, and showed the existence of 18 limit cycles. Very recently, Gouveia and Torregrosa [2020] found 24 limit cycles in a cubic switching polynomial system with degenerated Hopf and pseudo Hopf bifurcations, by perturbing a single Darboux center. Yu *et al.* [2021] constructed a cubic planar switching polynomial system with Z_2-symmetry, and proved that such a system could exhibit at least nine small-amplitude limit cycles around each of two symmetric foci, giving a total of 18 limit cycles.

If nonsmooth systems have different definitions for the continuous vector fields in two or more different regions divided by lines or curves, we call such systems switching systems. In this paper, we study switching planar systems, described by

$$(\dot{x}, \dot{y}) = \begin{cases} (\delta x - y + f^+(x,y), \\ \qquad x + \delta y + g^+(x,y)), \quad y > 0, \\ (\delta x - y + f^-(x,y), \\ \qquad x + \delta y + g^-(x,y)), \quad y < 0, \end{cases} \tag{1}$$

where $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ are analytic functions in x and y, starting from at least second-order terms. Actually, the origin of system (1) is an equilibrium. System (1) includes two subsystems: one is called upper system, defined for $y > 0$, and the other is called lower system, defined for $y < 0$.

To study the bifurcation of limit cycles associated with a singular point in a switching system, we need Lyapunov constants to determine the number and stability of bifurcating limit cycles. We will present a new version of Gasull–Torregrosa method [Gasull & Torregrosa, 2003] based on Poincaré return map [Andronov, 1973; Liu *et al.*, 2008] to compute the Lyapunov constants near the origin of the general system (1). Then we apply this method to study the center conditions and bifurcation of limit cycles in the following cubic switching system which has cubic homogeneous nonlinearities.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\delta(x - x^3) + b_0 y - (b_0 + 2)x^2 y + 2a_2 xy^2 + 2a_3 y^3 \\ -x + 2\delta y + x^3 + 2a_5 xy^2 + 2a_6 y^3 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} -\delta(x - x^3) + b_0 y - (b_0 + 2)x^2 y + 2b_2 xy^2 + 2b_3 y^3 \\ -x + 2\delta y + x^3 + 2b_5 xy^2 + 2b_6 y^3 \end{pmatrix}, & \text{if } y < 0, \end{cases} \tag{2}$$

where δ, a_i's and b_i's are real parameters, satisfying $|\delta| \ll 1$. Since our purpose is to find limit cycles as many as possible which bifurcate in system (2), we assume $b_0 > 0$ under which the origin is an elementary center. Obviously, the two points $(1, 0)$ and $(-1, 0)$ of system (2) are Hopf-type singular points. Note that the upper system of (2) is invariant under the change $(x, y) \to (-x, -y)$, and the lower system is also the same, which implies that the separate upper or lower system has Z_2-equivariant symmetry. However, we can obviously find that the whole system (2) is not Z_2-equivariant. Thus, when subsystems are equivariant, the whole system is not necessarily equivariant. Under the change $(x, y) \to (-x, -y)$, the system (2) becomes the following form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\delta(x - x^3) + b_0 y - (b_0 + 2)x^2 y + 2b_2 xy^2 + 2b_3 y^3 \\ -x + 2\delta y + x^3 + 2b_5 xy^2 + 2b_6 y^3 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} -\delta(x - x^3) + b_0 y - (b_0 + 2)x^2 y + 2a_2 xy^2 + 2a_3 y^3 \\ -x + 2\delta y + x^3 + 2a_5 xy^2 + 2a_6 y^3 \end{pmatrix}, & \text{if } y < 0. \end{cases} \tag{3}$$

Comparing (2) and (3), we know that the qualitative properties around $(-1, 0)$ in system (2) is the same as that around $(1, 0)$ in system (3). Actually, we can also obtain the system (3) by the substitution

$$(a_2, a_3, a_5, a_6, b_2, b_3, b_5, b_6) \to (b_2, b_3, b_5, b_6, a_2, a_3, a_5, a_6)$$

to system (2). Then we say that the qualitative properties around $(1, 0)$ in system (3) are the same as that around $(1, 0)$ in system (2). Now, we conclude that the qualitative properties around $(-1, 0)$ are the same as that around $(1, 0)$ in system (2). In other words, in system (2), if we find k limit cycles around $(1, 0)$, then we can also find k limit cycles around $(-1, 0)$; if $(1, 0)$ is a center, then $(-1, 0)$ can also be a center. However, note that the same dynamic behavior around the two points $(1, 0)$ and $(-1, 0)$ cannot exist at the same time. Based on the above description, we call such systems [like (2)] quasi-equivariant.

The main goal of this paper is to present a new version of the Gasull–Torregrosa method [Gasull & Torregrosa, 2003] for computing Lyapunov constants of switching systems. Based on the method, we derive center conditions and analyze the bifurcation of limit cycles in a quasi-equivariant cubic switching system with homogeneous nonlinearities. We first compute the first eight Lyapunov constants for the singular point $(1, 0)$ of system (2) to obtain the center conditions and prove the existence of seven limit cycles bifurcating from $(1, 0)$ or $(-1, 0)$, and one limit cycle bifurcating from $(0, 0)$. Then, we choose one of the center conditions with proper perturbations to construct a perturbed system, and compute the Lyapunov constants associated with the singular point of the perturbed system to prove the existence of ten limit cycles around $(1, 0)$, yielding that ten limit cycles can also bifurcate from $(-1, 0)$.

2. Computation of Lyapunov Constants

In this section, we present a new version of Gasull–Torregrosa method for solving center and limit cycle problems of switching systems. First, we introduce the Gasull–Torregrosa method which is used to compute the Lyapunov constant of switching system in many references [Tian & Yu, 2015; Li *et al.*, 2015; Guo *et al.*, 2018; Yu *et al.*, 2021]. The details of Gasull–Torregrosa's theory can be found in [Gasull & Torregrosa, 2003].

2.1. *Gasull–Torregrosa method*

Consider the general switching differential system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y + \sum_{k=2}^{n} X_k^+(x, y) \\ x + \delta y + \sum_{k=2}^{n} Y_k^+(x, y) \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} \delta x - y + \sum_{k=2}^{n} X_k^-(x, y) \\ x + \delta y + \sum_{k=2}^{n} Y_k^-(x, y) \end{pmatrix}, & \text{if } y < 0, \end{cases} \tag{4}$$

where $X_k^{\pm}(x,y)$ and $Y_k^{\pm}(x,y)$ are homogeneous polynomials in x and y. Under the polar coordinates transformation, $x = r\cos\theta$ and $y = r\sin\theta$, (4) can be rewritten as

$$\frac{dr}{d\theta} = \begin{cases} \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_k^{+}(\theta) r^k}{1 + \sum_{k=2}^{n} \Theta_k^{+}(\theta) r^{k-1}}, & \text{for } \theta \in (0,\pi), \\ \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_k^{-}(\theta) r^k}{1 + \sum_{k=2}^{n} \Theta_k^{-}(\theta) r^{k-1}}, & \text{for } \theta \in (-\pi,0), \end{cases} \tag{5}$$

where $\Upsilon_k^{\pm}(\theta)$ and $\Theta_k^{\pm}(\theta)$ are polynomials in $\sin\theta$ and $\cos\theta$ of degrees $k+1$. By the method of small parameters of Poincaré, the solutions of the upper and lower systems of (5) are given by

$$r^{+}(h,\theta) = \sum_{k\ge1} u_k(\theta) h^k, \quad r^{-}(h,\theta) = \sum_{k\ge1} v_k(\theta) h^k, \tag{6}$$

where $u_1(0) = v_1(0) = 1$, $u_k(0) = v_k(0) = 0$, $\forall\, k \ge 2$. Substituting the above solutions into (5), we can solve $u_k(\theta)$ or $v_k(\theta)$ one by one by integral operations. Consequently, we can define the following successive functions,

$$\triangle^{+}(h) = r^{+}(h,\pi) - h,$$

$$\triangle^{-}(h) = h - r^{-}(h,-\pi),$$

for the upper and lower systems of (5), respectively. Then, the successive function for the switching system (4) can be defined as

$$\begin{aligned} \triangle(h) &= \triangle^{+}(h) + \triangle^{-}(h) \\ &= r^{+}(h,\pi) - r^{-}(h,-\pi). \end{aligned} \tag{7}$$

It has been shown in [Gasull & Torregrosa, 2003] that the displacement function $\triangle(h)$ can be expanded as

$$\triangle(h) = \sum_{k=1}^{n} (u_k(\pi) - v_k(-\pi)) h^k = \sum_{k=0}^{n-1} V_k h^{k+1}, \tag{8}$$

where V_k is called the kth-order Lyapunov constant of the switching system (4).

2.2. *A new version of Gasull–Torregrosa method*

Both the above classical method and the method in [Tian & Yu, 2015; Li *et al.*, 2015; Guo *et al.*, 2018; Yu *et al.*, 2021] use integral operations to solve $u_k(\theta)$ or $v_k(\theta)$ one by one. Here, we replace the integral operations with algebraic computation, and give a new version.

Consider the equation

$$F(w,v,r) = \frac{A_3 + A_4 r}{1 + B_3 r + B_4 r^2}, \tag{9}$$

where A_k and B_k are homogeneous kth polynomials in w and v, $k = 3, 4$. We expand F into Taylor series at $r = 0$ of the form

$$F(w,v,r) = \sum_{i=0}^{\infty} F_i(w,v) r^i. \tag{10}$$

Then we have the following result.

Lemma 1. *The n-order partial derivative of (9) with respect to r has the following form*

$$\frac{\partial^n F(w,v,r)}{\partial r^n} = \frac{C_n \left(A_3 B_3^n + \sum_{i=0}^{n+1} Q_{n,i} r^i \right)}{(1 + B_3 r + B_4 r^2)^{n+1}}, \tag{11}$$

where $Q_{n,i}$ is the polynomial in w and v, C_n is a constant, $n = 1, 2, \ldots$. Assume that $Q_{n,0}$ does not contain the monomial like $A_3 B_3^n$. Then Eq. (11) satisfies the following conditions: $\deg(Q_{n,i}) < \deg(A_3 B_3^{n+i})$, $i = 0, 1, 2, \ldots, n+1$.

Proof. We prove this theorem by induction. For $n = 1$, we can obtain that

$$\frac{\partial F(w,v,r)}{\partial r} = -\frac{A_3 B_3 + A_4 B_4 r^2 + 2 A_3 B_4 r - A_4}{(1 + B_3 r + B_4 r^2)^2}. \tag{12}$$

Obviously, the first partial derivative of (9) satisfies our lemma. Now, we assume that the $(n-1)$th partial derivative of (9) satisfies the lemma, that is,

$$\frac{\partial^{n-1} F(w,v,r)}{\partial r^{n-1}} = \frac{C_{n-1} \left(A_3 B_3^{n-1} + \sum_{i=0}^{n} Q_{n-1,i} r^i \right)}{(1 + B_3 r + B_4 r^2)^n} \tag{13}$$

satisfying $\deg(Q_{n-1,i}) < \deg(A_3 B_3^{n+i-1})$, $i = 0, 1, 2, \ldots, n$. Here, $Q_{n-1,0}$ does not contain the monomial like $A_3 B_3^{n-1}$.

Then, solving the first partial derivative of (13) results in

$$\begin{aligned}
&\frac{\partial^n F(w,v,r)}{\partial r^n}\\
&= \frac{\partial}{\partial r}\left(\frac{\partial^{n-1}F(w,v,r)}{\partial r^{n-1}}\right)\\
&= C_{n-1}\frac{\left(\sum_{i=1}^{n} iQ_{n-1,i}r^{i-1}\right)(1+B_3r+B_4r^2) - n\left(A_3B_3^{n-1}+\sum_{i=0}^{n}Q_{n-1,i}r^i\right)(B_3+2B_4r)}{(1+B_3r+B_4r^2)^{n+1}}\\
&= C_{n-1}\frac{\left(\sum_{i=1}^{n} iQ_{n-1,i}r^{i-1}\right)(1+B_3r+B_4r^2) - n(B_3+2B_4r)\sum_{i=0}^{n}Q_{n-1,i}r^i - n(A_3B_3^n+2A_3B_3^{n-1}B_4r)}{(1+B_3r+B_4r^2)^{n+1}}\\
&= C_{n-1}\frac{-nA_3B_3^n+\left(\sum_{i=1}^{n} iQ_{n-1,i}r^{i-1}\right)(1+B_3r+B_4r^2) - n(B_3+2B_4r)\sum_{i=0}^{n}Q_{n-1,i}r^i - 2nA_3B_3^{n-1}B_4r}{(1+B_3r+B_4r^2)^{n+1}}.
\end{aligned} \tag{14}$$

Through simple calculation and arrangement, we find that

$$\frac{\partial^n F(w,v,r)}{\partial r^n} = \frac{C_n\left(A_3B_3^n+\sum_{i=0}^{n+1}Q_{n,i}r^i\right)}{(1+B_3r+B_4r^2)^{n+1}}, \tag{15}$$

where

$$\begin{aligned}
C_n &= -nC_{n-1},\\
Q_{n,0} &= -\frac{1}{n}(Q_{n-1,1}-nB_3Q_{n-1,0}),\\
Q_{n,1} &= -\frac{1}{n}[2Q_{n-1,2}+(1-n)B_3Q_{n-1,1}\\
&\quad - 2nA_3B_3^{n-1}B_4 - 2B_4nQ_{n-1,0}],\\
Q_{n,i} &= -\frac{1}{n}[(i+1)Q_{n-1,i+1}+(i-n)B_3Q_{n-1,i}\\
&\quad + (i-2n-1)B_4Q_{n-1,i-1}],\\
&\qquad \text{for } i=2,3,\ldots,n-1,\\
Q_{n,n} &= \frac{n+1}{n}B_4Q_{n-1,n-1},\\
Q_{n,n+1} &= B_4Q_{n-1,n}.
\end{aligned} \tag{16}$$

According to our hypothesis, we get the conclusion: $\deg(Q_{n,i}) < \deg(A_3B_3^{n+i})$, $i=0,1,2,\ldots,n+1$. ■

Now, we assume that the general switching system has the following form,

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{cases}\begin{pmatrix}\delta x - y + f_1(x,y)\\ x+\delta y + g_1(x,y)\end{pmatrix}, & \text{if } y>0,\\[2mm] \begin{pmatrix}\delta x - y + f_2(x,y)\\ x+\delta y + g_2(x,y)\end{pmatrix}, & \text{if } y<0.\end{cases} \tag{17}$$

By the polar coordinates transformation, $x = r\cos\theta$ and $y = r\sin\theta$, we can rewrite (17) as

$$\frac{dr}{d\theta} = \begin{cases}\dfrac{r^2\delta + rf_1\cos\theta + rg_1\sin\theta}{r+g_1\cos\theta - f_1\sin\theta}, & \text{for } \theta\in(0,\pi),\\[3mm] \dfrac{r^2\delta + rf_2\cos\theta + rg_2\sin\theta}{r+g_2\cos\theta - f_2\sin\theta}, & \text{for } \theta\in(-\pi,0),\end{cases} \tag{18}$$

where f_1, f_2, g_1 and g_2 are polynomials in $\sin\theta$, $\cos\theta$ and r. Next, we present how to solve $u_i(\theta)$ for upper system of (17). In the same way, we can get $v_i(\theta)$ for lower system of (17).

For the upper system of (17), substituting $\cos\theta = w$, $\sin\theta = v$ into the first half of (18) results in

$$F_u = \frac{dr}{d\theta} = \frac{r^2\delta + rwf_1 + rvg_1}{r+wg_1-vf_1}, \tag{19}$$

where f_1 and g_1 become functions of w, v and r.

In order to explain our method conveniently, we need to give the specific forms of f_1 and g_1 in the following. Because we only consider the cubic switching system in our paper, without loss of generality, we have

$$f_1 = f_{12}r^2 + f_{13}r^3, \quad g_1 = g_{12}r^2 + g_{13}r^3, \tag{20}$$

where f_{12} and g_{12} are quadratic homogeneous polynomials in w and v, and f_{13} and g_{13} are cubic homogeneous polynomials in w and v. Thus we can rewrite (19) as

$$F_u = \frac{r\delta + r^2[(wf_{12} + vg_{12}) + r(wf_{13} + vg_{13})]}{1 + (wg_{12} - vf_{12})r + (wg_{13} - vf_{13})r^2}. \tag{21}$$

In general, δ is a linear perturbation parameter and we can first let $\delta = 0$ to make the first Lyapunov constant equal to 0. Thus, by Lemma 1, when $\delta = 0$, F_u can be expanded in Taylor series at $r = 0$ to

$$F_u = r^2\left[\tilde{A}_3 + \sum_{n=1}^{\infty}\frac{1}{n!}\tilde{C}_n(\tilde{A}_3\tilde{B}_3^n + \tilde{Q}_{n,0})r^n\right], \tag{22}$$

where $\tilde{A}_3 = wf_{12} + vg_{12}$, $\tilde{B}_3 = wg_{12} - vf_{12}$, and $\tilde{Q}_{n,0}$ is the polynomial in w and v.

As described in classical theory, by the method of small parameters of Poincaré, the solution of the upper system of (17) is given by

$$r(h, \theta) = \sum_{p\geq 1} u_p(\theta)h^p, \tag{23}$$

where $u_1(0) = 1$, and $u_p(0) = 0$ for $\forall\, p \geq 2$. Substituting the above solution and $w = \cos\theta$, $v = \sin\theta$ into (19) and (22) yields that $u_1(\theta) = 1$ and $\frac{\mathrm{d}u_m(\theta)}{\mathrm{d}\theta} = F_{u,m}$, where $F_{u,m}$ has the form

$$F_{u,m} = \sum_{i_1 j_1 + \cdots + i_k j_k = m} \frac{[\tilde{C}_{i_1,\ldots,i_k,j_1,\ldots,j_k}(\tilde{A}_3\tilde{B}_3^{j_1+\cdots+j_k-2} + \tilde{Q}_{j_1+\cdots+j_k-2,0})u_{i_1}^{j_1}(\theta)u_{i_2}^{j_2}(\theta)\cdots u_{i_k}^{j_k}(\theta)]}{(j_1 + \cdots + j_k - 2)!}, \tag{24}$$

where $i_k \in \mathbb{N}_+$, $j_k \in \mathbb{N}$, $1 \leq i_1 < i_2 < \cdots < i_k < m$, $j_1 + \cdots + j_k \geq 2$, $\tilde{Q}_{0,0} = 0$. Here, $\tilde{A}_3$, $\tilde{B}_3, \tilde{Q}_{j_1+\cdots+j_k-2,0}$ become functions of $\cos\theta$ and $\sin\theta$, and $\tilde{C}_{i_1,\ldots,i_k,j_1,\ldots,j_k}$ is constant. Then, we have the lemma.

Lemma 2. *For the general cubic switching system, the maximal order of trigonometric functions* $\cos\theta$ *and* $\sin\theta$ *in the expression of* $u_m(\theta)$ *is equal to* $3(m-1)$.

Proof. Firstly, we denote that order(f) is the maximal order of $\cos\theta$ and $\sin\theta$ in function f. If $m = 1$, we have $u_1(\theta) = 1$. If $m = 2$, by (24), we have

$$\frac{du_2(\theta)}{d\theta} = \tilde{A}_3 u_1^2(\theta) = \tilde{A}_3.$$

Obviously, the lemma holds on for $u_1(\theta)$ and $u_2(\theta)$. By induction, we can assume that the lemma holds on for $u_3(\theta), \ldots, u_{m-1}(\theta)$. Next, we compute the order of $u_m(\theta)$. By Lemma 1, we can see in (24):

$$\begin{aligned}&\text{order}(\tilde{A}_3\tilde{B}_3^{j_1+\cdots+j_k-2})\\ &\quad = 3 + 3(j_1 + \cdots + j_k - 2) = 3\sum_{p=1}^{k} j_p - 3,\end{aligned}$$

$$\begin{aligned}&\text{order}(\tilde{Q}_{j_1+\cdots+j_k-2,0})\\ &\quad < \text{order}(\tilde{A}_3\tilde{B}_3^{j_1+\cdots+j_k-2}) = 3\sum_{p=1}^{k} j_p - 3,\end{aligned}$$

$$\begin{aligned}&\text{order}(u_{i_1}^{j_1}(\theta)u_{i_2}^{j_2}(\theta)\cdots u_{i_k}^{j_k}(\theta))\\ &\quad = \sum_{p=1}^{k} 3(i_p - 1)j_p = 3\sum_{p=1}^{k} i_p j_p - 3\sum_{p=1}^{k} j_p\\ &\quad = 3m - 3\sum_{p=1}^{k} j_p.\end{aligned}$$

Thus,

$$\begin{aligned}\text{order}\left(\frac{\mathrm{d}u_m(\theta)}{\mathrm{d}\theta}\right) &= 3\sum_{p=1}^{k} j_p - 3 + 3m - 3\sum_{p=1}^{k} j_p\\ &= 3(m-1).\end{aligned}$$

Because order means the maximal order of trigonometric functions $\cos\theta$ and $\sin\theta$, we can get the conclusion that order$(u_m(\theta)) = 3(m-1)$. ■

Through the above analysis, we have the following theorem.

Theorem 1. *The form of* $u_q(\theta)$ *is*:

$$
\begin{aligned}
&G_q(\theta) = G_{q,0} + G_{q,1}(\theta) + G_{q,2}(\theta), \quad G_{q,0} \text{ is constant term},\\
&G_{q,1}(\theta) = \sum_{q_2=1}^{d_{q,2}} \theta^{q_2} \left\{ \sum_{q_1=1}^{d_{q,1}} [S_{1,q,q_1,q_2}\cos(q_1\theta) + S_{2,q,q_1,q_2}\sin(q_1\theta)] + S_{3,0,q,q_2} \right\} + S_{4,0,q,d_{q,2}+1}\theta^{d_{q,2}+1},\\
&G_{q,2}(\theta) = \sum_{q_3=1}^{d_{q,1}} [T_{1,q,q_3}\sin(q_3\theta) + T_{2,q,q_3}\cos(q_3\theta)],
\end{aligned}
\tag{25}
$$

where S_{1,q,q_1,q_2}, S_{2,q,q_1,q_2}, $S_{3,0,q,q_2}$, $S_{4,0,q,d_{q,2}+1}$, T_{1,q,q_3} *and* T_{2,q,q_3} *are all constants*, $d_{q,1}$ *is the order of* $\cos(\theta)$ *and* $\sin(\theta)$ *in* $F_{u,q}$, $d_{q,2}$ *is the degree of* θ *in* $F_{u,q}$.

Note that in the above theorem, for cubic switching system, we know that $d_{q,1} = 3(q-1)$ for $u_q(\theta)$ by Lemma 2. In order to get $u_q(\theta)$, we just need to substitute (25) to (24) and compare the coefficients of $\cos(i\theta)$, $\sin(i\theta)$, $i = 1, \dots, d_{q,1}$ and θ on both sides to determine the values of $G_{q,0}$, S_{1,q,q_1,q_2}, S_{2,q,q_1,q_2}, $S_{3,0,q,q_2}$, $S_{4,0,q,d_{q,2}+1}$, T_{1,q,q_3} and T_{2,q,q_3}.

Algorithm 1 Algorithm for solving u_q.

Input: $f_1(x,y)$, $g_1(x,y)$

Output: $u_q(\theta)$

$d_{q,1} := \deg(F_{u,q}, [\cos(\theta), \sin(\theta)])$,

$d_{q,2} := \deg(F_{u,q}, \theta)$,

$G_{q,0} := solve(G_q(0) = 0, G_{q,0})$,

$\overline{F}_{u,q} := combine(F_{u,q}, trig)$,

$H_q(\theta) = \frac{dG_q(\theta)}{d\theta} - \overline{F}_{u,q}$,

$\overline{H}_q(\theta) = combine(H_q(\theta), trig)$,

Solve all undetermined coefficients in (25) from the equation $\overline{H}_q(0) = 0$.

From (8), we know that the origin is a center of system (4) if and only if $\triangle(h) \equiv 0$ for $0 < h \ll 1$. The isolated zeros of $\triangle(h) = 0$ near $h = 0$ correspond to the number of limit cycles around the origin. By simple computation, we get $u_1(\theta) = v_1(\theta) = e^{\delta\theta}$, yielding $V_0 = \frac{1}{e^{\pi\delta}}(e^{2\pi\delta} - 1)$. Then we have $V_0 = 0$ if and only if $\delta = 0$. As we all know, k must be odd for V_k of smooth systems [Han & Yu, 2012]. In general, $V_k \neq 0$ with k being any positive integer for the switching system (4). Next, we turn to discuss how to determine the maximal number of limit cycles which may bifurcate from a Hopf critical point. Generally, the following theorem gives sufficient conditions for the existence of small-amplitude limit cycles in the switching system (4). (The proof can be found in [Tian & Yu, 2015].)

Theorem 2 [Tian & Yu, 2015]. *Suppose that there exists a sequence of Lyapunov constants of system (4),* $V_{i_0}, V_{i_1}, \dots, V_{i_k}$, *with* $1 = i_0 < i_1 < \cdots < i_k$, *such that* $V_j = O(|V_{i_0}, \dots, V_{i_l}|)$ *for any* $i_l < j < i_{l+1}$. *Further, if at the critical point* $\boldsymbol{C}$, $V_{i_0} = V_{i_1} = \cdots = V_{i_k-1} = 0$, $V_{i_k} \neq 0$, *and*

$$
\det\left[\frac{\partial(V_{i_0}, V_{i_1}, \dots, V_{i_{k-1}})}{\partial(c_1, c_2, \dots, c_k)}\right]_{\mathbf{C}} \neq 0, \tag{26}
$$

then system (4) has exactly k limit cycles in a δ-ball with its center at the origin.

Compared with the smooth system, the center problem in switching systems is more complicated. In order to prove the center conditions for system (4), we have the following lemmas.

Lemma 3 [Chen & Zhang, 2012]. *If the upper and lower systems of (4) have the first integrals* $H^+(x,y)$ *and* $H^-(x,y)$ *near the origin, respectively, and either* $H^+(x,y)$ *or* $H^-(x,y)$ *is an even function in* x *or* $H^+(x,0) \equiv H^-(x,0)$, *then the origin of system (4) is a center.*

Lemma 4 [Li *et al.*, 2015]. *Assume that* $\delta = 0$. *If system (4) is symmetric with respect to the x-axis, i.e. the functions on the right-hand side of system (4) satisfy*

$$X_k^+(x,y) = -X_k^-(x,-y),$$
$$Y_k^+(x,y) = Y_k^-(x,-y),$$

or if system (4) is symmetric with respect to the y-axis, i.e. the functions on the right-hand side of system (4) satisfy

$$X_k^+(x,y) = X_k^+(-x,y),$$
$$X_k^-(x,y) = X_k^-(-x,y),$$
$$Y_k^+(x,y) = -Y_k^+(-x,y),$$
$$Y_k^-(x,y) = -Y_k^-(-x,y),$$

then the origin of system (4) is a center.

3. Center Conditions and Hopf Bifurcation

In this section, we consider the center conditions and bifurcation of limit cycles for the switching cubic system (2). Because the qualitative properties around $(-1,0)$ are the same as that around $(1,0)$ in the system (2), hence we only need to consider the center conditions and Hopf bifurcation at the singular point $(1,0)$.

In order to study the center conditions and limit cycles bifurcation around the Hopf critical point $(1,0)$, we need to compute its Lyapunov constants associated with the Hopf critical point. To achieve this, we introduce the following transformation,

$$x = x_1 + 1, \quad y = y_1, \quad t \to \frac{1}{2}t,$$

into system (2) to obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x_1 - y_1 + \frac{3}{2}\delta x_1^2 - (b_0+2)x_1y_1 + a_2y_1^2 + \frac{1}{2}\delta x_1^3 \\ -\left(\frac{1}{2}b_0+1\right)x_1^2y_1 + a_2x_1y_1^2 + a_3y_1^3 \\ x_1 + \delta y_1 + \frac{3}{2}x_1^2 + a_5y_1^2 + \frac{1}{2}x_1^3 + a_5x_1y_1^2 + a_6y_1^3 \end{pmatrix}, & \text{if } y_1 > 0, \\ \begin{pmatrix} \delta x_1 - y_1 + \frac{3}{2}\delta x_1^2 - (b_0+2)x_1y_1 + b_2y_1^2 + \frac{1}{2}\delta x_1^3 \\ -\left(\frac{1}{2}b_0+1\right)x_1^2y_1 + b_2x_1y_1^2 + b_3y_1^3 \\ x_1 + \delta y_1 + \frac{3}{2}x_1^2 + b_5y_1^2 + \frac{1}{2}x_1^3 + b_5x_1y_1^2 + b_6y_1^3 \end{pmatrix}, & \text{if } y_1 < 0. \end{cases} \tag{27}$$

Clearly, the singular point $(1,0)$ of system (2) corresponds to the origin of (27), which is a Hopf-type critical point. In the following, we will use our method in the previous section to compute the Lyapunov constants for the origin of system (27), and use them to derive the center conditions and to consider limit cycle bifurcation.

3.1. *Center conditions for system (27)*

With the aid of the program in Maple, we have computed the Lyapunov constants associated with the singular points $(1,0)$ of system (2), as given in the following theorem.

Theorem 3. *For system (27), the first four Lyapunov constants at the origin are given by*

$$V_0 = \frac{1}{e^{\pi\delta}}(e^{2\pi\delta} - 1),$$

$$V_1 = \frac{4}{3}(a_5 - b_5),$$

$$V_2 = -\frac{\pi}{8}[(a_2+b_2)(1+b_0-2b_5) - 3(a_6+b_6)],$$

$$V_3 = \frac{4}{45}\{(2+b_0-2b_5)[4b_2(a_2+b_2) - 3(a_3-b_3)] - 4(b_2+3b_6)(a_2+b_2)\}.$$

For higher Lyapunov constants, there are several cases listed below.

(I) *If* $2 + b_0 - 2b_5 \neq 0,$ *then*

$$V_4 = -\frac{(a_2 + b_2)\pi}{288(2 + b_0 - 2b_5)}\{(2 + b_0 - 2b_5)^2[6b_0(3 + b_0 + 3b_5) + 5b_2(a_2 + b_2) + 30b_3] - 5(2 + b_0 - 2b_5)(a_2 - 5b_2)(b_2 + 3b_6) - 30(b_2 + 3b_6)^2\},$$

$$V_5 = -\frac{32(a_2 + b_2)}{14175}\{(2 + b_0 - 2b_5)[9(a_2 + b_2)b_0^2 - 18b_0(b_2b_5 + a_2b_5 - 2b_2 + 3b_6 + a_2) + 10b_2(a_2 + b_2)^2] - 10(a_2 + b_2)^2(b_2 + 3b_6)\}.$$

($\mathrm{I_a}$) *Further, if* $5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5) \neq 0,$ *then*

$$V_6 = \frac{b_0(a_2 + b_2)(2 - b_0 + 2b_5)(2 + b_0 - 2b_5)\pi}{92160[5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5)]^2}F_1,$$

$$V_7 = \frac{b_0(a_2 + b_2)(-2 + b_0 - 2b_5)(2 + b_0 - 2b_5)}{1219276800[5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5)]^3}F_2;$$

($\mathrm{I_b}$) *or if* $5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5) = 0,$ *then*

$$V_5 = -\frac{32}{25515b_0}[(a_2 - b_2)(a2 + b2)^3][5(a_2 + b_2)^2 + 36b_0],$$

$$V_6 = -\frac{\pi}{933120b_0b_2}[15309(b_2 + 3b_6)^2b_0^3 + 135b_2^2(443b_2^2 + 378b_2b_6 - 1701b_6^2)b_0^2 + 600b_2^5(289b_2 - 567b_6)b_0 - 14000b_2^8],$$

$$V_7 = \frac{1}{4115059200b_0^2b_2}\{32768b_0(20b_2^3 + 9b_0b_2 + 27b_0b_6)[567(b_2 + 3b_6)^2b_0^2 - 40b_2^3(89b_2 - 189b_6)b_0 + 560b_2^6] - 735\pi(40b_2^2 + 27b_0)[15309(b_2 + 3b_6)^2b_0^3 + 135b_2^2(443b_2^2 + 378b_2b_6 - 1701b_6^2)b_0^2 + 600b_2^5(289b_2 - 567b_6)b_0 - 14000b_2^8]\}.$$

(II) *If* $2 + b_0 - 2b_5 = 0,$ *then*

$$V_3 = -\frac{16}{45}(a_2 + b_2)(b_2 + 3b_6), \quad V_4 = \frac{5\pi}{64}(-a_3 + b_3)(b_2 + 3b_6), \quad V_5 = V_6 = V_7 = 0.$$

In the above expressions of V_k's, we have set $V_0 = V_1 = V_2 = \cdots = V_{k-1} = 0$ in computing V_k, for $k = 1, 2, 3, \ldots, 7$. Here,

$$F_1 = 8[12b_0 + 35(a_2 + b_2)^2][9b_0(2 + b_0 - 2b_5) + 5(a_2 + b_2)^2]^2 + 15120(a_2 - b_2)^2[9b_0(2 + b_0 - 2b_5) + 5(a_2 + b_2)^2] - 315(a_2 - b_2)^2\{24[5(a_2 + b_2)^2 - 36b_5 + 36]b_0 + 5[5(a_2 + b_2)^2 + 48](a_2 + b_2)^2\}$$

and

$$F_2 = [5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5)]\{[5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5)]\{[5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5)] \times [524288(b_2 - a_2)(12b_0 + 35(a_2 + b_2)^2) + \pi(3810240b_0^2 + 317520(35(a_2 + b_2)^2 - 24b_5 + 6)b_0$$

$$+\,5556600(a_2+b_2)^2(1-4b_5))]-3932160(a_2-b_2)[76(a_2+b_2)^2b_0+252(a_2-b_2)^2+7(a_2+b_2)^4]$$
$$+\,33339600\pi(a_2-b_2)^2[18b_0-5(a_2+b_2)^2-27]\}+1890(a_2-b_2)^2\{262144(a_2-b_2)[5(a_2+b_2)^2$$
$$-\,36b_5+36]-6615\pi[25(a_2+b_2)^4+60(a_2+b_2)^2+1296b_5-1296]\}b_0+1575(a_2-b_2)^2(a_2+b_2)^2$$
$$\times\{65536(a_2-b_2)[5(a_2+b_2)^2+96]+6615\pi[5(a_2+b_2)^2(12b_5+13)+432]\}\}$$
$$+\,34406400(a_2+b_2)^2(a_2-b_2)^3\{72[5(a_2+b_2)^2+36b_5-36]b_0+5(a_2+b_2)^2[5(a_2+b_2)^2-144]\}.$$

Now, we turn to discuss the center conditions of system (27). From Theorem 3 we have the following result.

Theorem 4. *System (27) has a center at the origin if and only if* $\delta = 0$ *and one of the following conditions is satisfied*:

(I) $a_5 - b_5 = 2 - b_0 + 2b_5 = a_2 - a_6 = a_3 - b_3 = b_0^2 + 2b_3 = b_2 - b_6 = 0$,
(II) $a_5 - b_5 = a_6 + b_6 = a_3 - b_3 = a_2 + b_2 = 0$,
(III) $a_5 - b_5 = 2 + b_0 - 2b_5 = b_2 + 3b_6 = a_2 + 3a_6 = 0$.

Proof. To prove that we have obtained a complete classification on the center conditions, we first prove the necessity of the conditions in Theorem 4. Assume that system (27) has a center at the origin, then all the Lyapunov constants should vanish. From Theorem 3, $V_0 = 0$ yields $\delta = 0$. Then, we use a_5 and a_6 to linearly solve $V_1 = 0$ and $V_2 = 0$, respectively. In the case $2 + b_0 - 2b_5 \neq 0$, we use a_3 to linearly solve $V_3 = 0$. Then, if we use the condition $a_2 + b_2 = 0$ to solve $V_4 = 0$ and $V_5 = 0$, we have the center condition (II). Otherwise, we linearly solve $V_4 = 0$ using b_3. Further, suppose that $5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5) \neq 0$, we solve $V_5 = 0$ using b_6. Next, if we use the condition $2 - b_0 + 2b_5 = 0$ to solve $V_6 = 0$ and $V_7 = 0$, then we get the center condition (I). Otherwise, we cannot get center conditions, and this case will be discussed in the next subsection. However, if we assume $5(a_2 + b_2)^2 + 9b_0(2 + b_0 - 2b_5) = 0$, by a direct computation, we find that all the Lyapunov constants cannot vanish. In the case $2 + b_0 - 2b_5 = 0$, by solving $V_3 = 0$, we obtain two conditions $a_2 + b_2 = 0$ and $b_2 + 3b_6 = 0$, which yield the center conditions (II) and (III), respectively.

Next, we prove the sufficiency of the conditions.

When condition (I) holds with $\delta = 0$, system (27) becomes

$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 - (b_0+2)x_1y_1 + a_2y_1^2 - \left(\dfrac{1}{2}b_0 + 1\right)x_1^2y_1 + a_2x_1y_1^2 - \dfrac{1}{2}b_0^2y_1^3, \\[2ex] \dfrac{dy_1}{dt} = x_1 + \dfrac{3}{2}x_1^2 + \left(\dfrac{1}{2}b_0 - 1\right)y_1^2 + \dfrac{1}{2}x_1^3 + \left(\dfrac{1}{2}b_0 - 1\right)x_1y_1^2 + a_2y_1^3, \end{cases} \quad (y_1 > 0);$$
$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 - (b_0+2)x_1y_1 + b_2y_1^2 - \left(\dfrac{1}{2}b_0 + 1\right)x_1^2y_1 + b_2x_1y_1^2 - \dfrac{1}{2}b_0^2y_1^3, \\[2ex] \dfrac{dy_1}{dt} = x_1 + \dfrac{3}{2}x_1^2 + \left(\dfrac{1}{2}b_0 - 1\right)y_1^2 + \dfrac{1}{2}x_1^3 + \left(\dfrac{1}{2}b_0 - 1\right)x_1y_1^2 + b_2y_1^3, \end{cases} \quad (y_1 < 0). \tag{28}$$

The upper and lower systems respectively have the analytic first integrals,

$$H_1(x_1, y_1) = \frac{1}{4}\ln[(x_1+1)^2 + b_0y_1^2] - \frac{1}{2b_0[(x_1+1)^2 + b_0y_1^2]}$$
$$\times\left\{\frac{a_2(x_1+1)}{\sqrt{b_0}|x_1+1|}[(x_1+1)^2 + b_0y_1^2]\tan^{-1}\frac{b_0y_1}{\sqrt{b_0}|x_1+1|} + \left[(x_1+1)^2 - a_2y_1(x_1+1) - \frac{b_0}{2}\right]\right\}$$

and

$$H_2(x_1, y_1) = \frac{1}{4}\ln[(x_1+1)^2 + b_0 y_1^2] - \frac{1}{2b_0[(x_1+1)^2 + b_0 y_1^2]} \times \left\{ \frac{b_2(x_1+1)}{\sqrt{b_0}|x_1+1|}[(x_1+1)^2 + b_0 y_1^2]\tan^{-1}\frac{b_0 y_1}{\sqrt{b_0}|x_1+1|} + \left[(x_1+1)^2 - b_2 y_1(x_1+1) - \frac{b_0}{2}\right]\right\},$$

showing that $H_1(x_1, 0) = H_2(x_1, 0)$. By Lemma 3, the origin of system (27) is a center.

When the condition (II) holds with $\delta = 0$, system (27) is reduced to

$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 - (b_0+2)x_1y_1 - b_2y_1^2 - \left(\dfrac{1}{2}b_0 + 1\right)x_1^2y_1 - b_2x_1y_1^2 + b_3y_1^3, \\ \dfrac{dy_1}{dt} = x_1 + \dfrac{3}{2}x_1^2 + b_5y_1^2 + \dfrac{1}{2}x_1^3 + b_5x_1y_1^2 - b_6y_1^3, \end{cases} \quad (y_1 > 0);$$
$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 - (b_0+2)x_1y_1 + b_2y_1^2 - \left(\dfrac{1}{2}b_0 + 1\right)x_1^2y_1 + b_2x_1y_1^2 + b_3y_1^3, \\ \dfrac{dy_1}{dt} = x_1 + \dfrac{3}{2}x_1^2 + b_5y_1^2 + \dfrac{1}{2}x_1^3 + b_5x_1y_1^2 + b_6y_1^3, \end{cases} \quad (y_1 < 0), \tag{29}$$

showing that the system is symmetric with the x_1-axis, and thus by Lemma 4, the origin of system (27) is a center.

When the condition (III) is satisfied with $\delta = 0$, system (27) becomes

$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 - (b_0+2)x_1y_1 - 3a_6y_1^2 - \left(\dfrac{1}{2}b_0 + 1\right)x_1^2y_1 - 3a_6x_1y_1^2 + a_3y_1^3, \\ \dfrac{dy_1}{dt} = x_1 + \dfrac{3}{2}x_1^2 + \left(\dfrac{1}{2}b_0 + 1\right)y_1^2 + \dfrac{1}{2}x_1^3 + \left(\dfrac{1}{2}b_0 + 1\right)x_1y_1^2 + a_6y_1^3, \end{cases} \quad (y_1 > 0);$$
$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 - (b_0+2)x_1y_1 - 3b_6y_1^2 - \left(\dfrac{1}{2}b_0 + 1\right)x_1^2y_1 - 3b_6x_1y_1^2 + b_3y_1^3, \\ \dfrac{dy_1}{dt} = x_1 + \dfrac{3}{2}x_1^2 + \left(\dfrac{1}{2}b_0 + 1\right)y_1^2 + \dfrac{1}{2}x_1^3 + \left(\dfrac{1}{2}b_0 + 1\right)x_1y_1^2 + b_6y_1^3, \end{cases} \quad (y_1 < 0). \tag{30}$$

The upper and lower systems have analytic first integrals,

$$H_{11}(x_1, y_1) = -\frac{1}{2}(x_1^2 + y_1^2) - \left(1 + \frac{b_0}{2}\right)\left(1 + \frac{1}{2}x_1\right)x_1y_1^2 - a_6y_1^3(x_1+1) - \frac{1}{2}x_1^3\left(1 + \frac{1}{4}x_1\right) + \frac{1}{4}a_3y_1^4$$

and

$$H_{22}(x_1, y_1) = -\frac{1}{2}(x_1^2 + y_1^2) - \left(1 + \frac{b_0}{2}\right)\left(1 + \frac{1}{2}x_1\right)x_1y_1^2 - b_6y_1^3(x_1+1) - \frac{1}{2}x_1^3\left(1 + \frac{1}{4}x_1\right) + \frac{1}{4}b_3y_1^4,$$

respectively, indicating that $H_{11}(x_1, 0) = H_{22}(x_1, 0)$. By Lemma 3, the origin of system (27) is a center.

The proof is completed. ■

3.2. *Bifurcation of limit cycles in system (2)*

In this section, we study the bifurcation of limit cycles in system (2). Consider the limit cycles bifurcating from the origin $(0,0)$, $(1,0)$ or $(-1,0)$. The following result directly follows Theorem 3.

Theorem 5. *System (2) can have eight limit cycles with the* $7 \cup 1$ *distribution around the singular points* $(1,0) \cup (0,0)$ *or* $(-1,0) \cup (0,0)$.

Proof. For system (27), as discussed in the proof of Theorem 3, we set $\delta = 0$ to get $V_0 = 0$. From the second Lyapunov constant V_1 in Theorem 3, we solve $V_1 = 0$ to obtain $a_5 = b_5$. Then, solving $V_2 = 0$ yields $a_6 = \frac{1}{3}[(a_2+b_2)(1+b_0-2b_5)-3b_6]$. In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin of (27), we assume that

$$\begin{aligned}(a_2^2 - b_2^2)&[(b_0-2b_5)^2-4](3b_0-6b_5+10)\\ &\times [5(a_2+b_2)^2 + 9b_0(2+b_0-2b_5)]\\ &\times [9b_0(2+b_0-2b_5)+20b_2^2] \neq 0.\end{aligned}$$

Then, we solve

$$V_3 = V_4 = V_5 = 0$$

to obtain

$$\begin{aligned}a_3 &= \frac{1}{3(2+b_0-2b_5)}\{(2+b_0-2b_5)\\ &\quad\times [4b_2(a_2+b_2)+3b_3] - 4(b_2+3b_6)(a_2+b_2)\},\\ b_3 &= \frac{1}{30(2+b_0-2b_5)^2}\{5(2+b_0-2b_5)(a_2-5b_2)\\ &\quad\times (b_2+3b_6) + 30(b_2+3b_6)^2 - (2+b_0-2b_5)^2\\ &\quad\times [6b_0(3+b_0+3b_5)+5b_2(a_2+b_2)]\},\\ b_6 &= \frac{1}{6[5(a_2+b_2)^2+9b_0(2+b_0-2b_5)]}\\ &\quad\times\{(2+b_0-2b_5)[9(a_2+b_2)b_0^2\\ &\quad - 18b_0(b_2b_5+a_2b_5-2b_2+a_2)\\ &\quad + 10b_2(a_2+b_2)^2] - 10(a_2+b_2)^2b_2\}.\end{aligned}$$

To simplify the expressions of V_6 and V_7, with the aid of Maple built-in command "rem" we obtain the remainder equation,

$$F_{21} = \text{rem}(F_2, F_1, b_5) = \frac{17203200(a_2+b_2)^2(a_2-b_2)^3[5(a_2+b_2)^2+36b_0]}{35(a_2+b_2)^2+12b_0}F_{21a},$$

where

$$F_{21a} = 35(a_2+b_2)^4 - 24(a_2+b_2)^2b_0 + 9072b_0^2.$$

It is easy to verify that $F_{21a} = 0$ has no real solutions, which implies that we can obtain at most seven small-amplitude limit cycles around the origin of (27).

With the results obtained above, a direct calculation shows that the determinant evaluated at the critical values is given by

$$\det\left[\frac{\partial(V_1,V_2,V_3,V_4,V_5,V_6)}{\partial(a_5,a_6,a_3,b_3,b_6,b_5)}\right] = -\frac{(b_0-2b_5+2)^2b_0(a_2+b_2)^3\pi^3}{122472000[5(a_2+b_2)^2+9b_0(2+b_0-2b_5)]^2}F_{\text{det}},$$

where F_{det} can be found in the supplement posted on the journal website. Then, we obtain

$$\begin{aligned}\text{resultant}(F_1, F_{\text{det}}, a_2) &= 1.14829757638306762275116811878 4\times 10^{56}(b_0-2b_5+2)^6(b_0-2b_5-2)^{10}\\ &\quad\times b_0^{18}(3b_0-6b_5+10)^4(35b_2^2+3b_0)^2[9b_0(2+b_0-2b_5)+20b_2^2]^4 \neq 0,\end{aligned}$$

implying, by Theorem 2, that system (27) can indeed have seven small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Thus, in system (2), seven limit cycles can exist around the singular point $(1,0)$.

Moreover, under the above conditions we prove that there is one limit cycle bifurcating from the origin of system (2). To show this, with the transformation,

$$x = \sqrt{b_0}x_2, \quad y = y_2, \quad t \to \frac{-1}{\sqrt{b_0}}t,$$

when $\delta = 0$, system (2) is transformed to the following form,

$$\begin{cases} \dfrac{dx_2}{dt} = -y_2 + (b_0+2)x_2^2y_2 - \dfrac{2a_2}{\sqrt{b_0}}x_2y_2^2 - \dfrac{2a_3}{b_0}y_2^3, \\ \dfrac{dy_2}{dt} = x_2 - b_0x_2^3 - 2a_5x_2y_2^2 - \dfrac{2a_6}{\sqrt{b_0}}y_2^3, \end{cases} \quad (y_2 > 0);$$
$$\begin{cases} \dfrac{dx_2}{dt} = -y_2 + (b_0+2)x_2^2y_2 - \dfrac{2b_2}{\sqrt{b_0}}x_2y_2^2 - \dfrac{2b_3}{b_0}y_2^3, \\ \dfrac{dy_2}{dt} = x_2 - b_0x_2^3 - 2b_5x_2y_2^2 - \dfrac{2b_6}{\sqrt{b_0}}y_2^3. \end{cases} \quad (y_2 < 0). \tag{31}$$

It can be shown that the first Lyapunov constant evaluated at the origin of (31) is given by

$$U_1 = -\frac{(a_2+3a_6+b_2+3b_6)\pi}{4\sqrt{b_0}}.$$

It is easy to verify that if system (2) has seven limit cycles around $(1, 0)$, then

$$U_1 = -\frac{(b_0-2b_5+2)(a_2+b_2)\pi}{4\sqrt{b_0}} \neq 0,$$

implying that system (2) can have one limit cycle around the origin $(0, 0)$. Hence, system (2) can have at least eight limit cycles with the $7 \cup 1$ distribution around the singular points $(1, 0) \cup (0, 0)$. In the same way, system (2) can have at least eight limit cycles with the $7 \cup 1$ distribution around the singular points $(-1, 0) \cup (0, 0)$. ■

4. Bifurcation of Limit Cycles by Perturbation

In this section, we want to obtain small-amplitude limit cycles around the center $(1, 0)$ or $(-1, 0)$ by perturbing the system (2). In view of the fact that system (2) has more free parameters under center condition (II) than that under the center condition (I) or (III), we may use the center condition (II) to generate more limit cycles. Thus, we add cubic homogeneous perturbations to system (2) under center condition (II) to obtain the following perturbed system:

$$\begin{cases} \dfrac{dx_2}{dt} = b_0y_2 - (b_0+2)x_2^2y_2 - 2b_2x_2y_2^2 + 2b_3y_2^3 \\ \qquad - \epsilon[\delta(x_2^3 - x_2) + p_8y_2 + p_7(x_2^3 - x_2) + p_1x_2^2y_2 + p_2x_2y_2^2 + p_3y_2^3], \\ \dfrac{dy_2}{dt} = -x_2 + x_2^3 + 2b_5x_2y_2^2 - 2b_6y_2^3 \\ \qquad - \epsilon[2\delta y_2 + p_{10}y_2 + p_9(x_2^3 - x_2) + p_4x_2^2y_2 + p_5x_2y_2^2 + p_6y_2^3], \end{cases} \quad (y_2 > 0);$$
$$\begin{cases} \dfrac{dx_2}{dt} = b_0y_2 - (b_0+2)x_2^2y_2 + 2b_2x_2y_2^2 + 2b_3y_2^3 \\ \qquad - \epsilon[\delta(x_2^3 - x_2) + q_8y_2 + q_7(x_2^3 - x_2) + q_1x_2^2y_2 + q_2x_2y_2^2 + q_3y_2^3], \\ \dfrac{dy_2}{dt} = -x_2 + x_2^3 + 2b_5x_2y_2^2 + 2b_6y_2^3 \\ \qquad - \epsilon[2\delta y_2 + q_{10}y_2 + q_9(x_2^3 - x_2) + q_4x_2^2y_2 + q_5x_2y_2^2 + q_6y_2^3], \end{cases} \quad (y_2 < 0), \tag{32}$$

where δ, p_i's and q_i's are real parameters, satisfying $|\delta| \ll 1$ and $0 < \epsilon \ll 1$.

In order for system (32) to have Hopf singular points at $(1, 0)$, we set $p_{10} = -p_4 - 2p_7$, $q_{10} = -q_4 - 2q_7$. By a direct computation, we can show that a further simplification in computation can be made by setting $p_7 = p_9 = q_7 = q_9 = 0$, $p_8 = -p_1$, $q_8 = -q_1$ and $b_0 = 1$.

Theorem 6. *The perturbed system (32) can have at least ten small-amplitude limit cycles around the singular points* $(1,0)$ *or* $(-1,0)$.

Proof. Based on the form of (32), we say that the qualitative properties around $(-1,0)$ are the same with that around $(1,0)$ in system (32). Hence we only need to consider the Hopf bifurcation at the singular point $(1,0)$. We need to compute its Lyapunov constants to study the limit cycles bifurcating from the Hopf critical point $(1,0)$. Now, we give the following transformation,

$$x_2 = 1 - X, \quad y_2 = Y, \quad t \to -\frac{1}{2}t,$$

into system (32), so the singular point $(1,0)$ of (32) becomes the origin of the following system,

$$\begin{cases} \dfrac{dX}{dt} = \epsilon\delta X - Y - \dfrac{3}{2}\epsilon\delta X^2 + (\epsilon p_1 + 3)XY - \left(b_2 + \dfrac{1}{2}\epsilon p_2\right)Y^2 + \dfrac{1}{2}\epsilon\delta X^3 \\ \qquad - \left(\dfrac{3}{2} + \dfrac{1}{2}\epsilon p_1\right)X^2Y + \left(b_2 + \dfrac{1}{2}\epsilon p_2\right)XY^2 + \left(b_3 - \dfrac{1}{2}\epsilon p_3\right)Y^3, \\ \dfrac{dY}{dt} = X + \epsilon\delta Y - \dfrac{3}{2}X^2 - \epsilon p_4 XY - \left(b_5 - \dfrac{1}{2}\epsilon p_5\right)Y^2 + \dfrac{1}{2}X^3 + \dfrac{1}{2}\epsilon p_4 X^2 Y \\ \qquad + \left(b_5 - \dfrac{1}{2}\epsilon p_5\right)XY^2 + \left(b_6 + \dfrac{1}{2}\epsilon p_6\right)Y^3, \end{cases} \quad (Y > 0);$$
$$\begin{cases} \dfrac{dX}{dt} = \epsilon\delta X - Y - \dfrac{3}{2}\epsilon\delta X^2 + (\epsilon q_1 + 3)XY + \left(b_2 - \dfrac{1}{2}\epsilon q_2\right)Y^2 + \dfrac{1}{2}\epsilon\delta X^3 \\ \qquad - \left(\dfrac{3}{2} + \dfrac{1}{2}\epsilon q_1\right)X^2Y - \left(b_2 - \dfrac{1}{2}\epsilon q_2\right)XY^2 + \left(b_3 - \dfrac{1}{2}\epsilon q_3\right)Y^3, \\ \dfrac{dY}{dt} = X + \epsilon\delta Y - \dfrac{3}{2}X^2 - \epsilon q_4 XY - \left(b_5 - \dfrac{1}{2}\epsilon q_5\right)Y^2 + \dfrac{1}{2}X^3 + \dfrac{1}{2}\epsilon q_4 X^2 Y \\ \qquad + \left(b_5 - \dfrac{1}{2}\epsilon q_5\right)XY^2 - \left(b_6 - \dfrac{1}{2}\epsilon q_6\right)Y^3. \end{cases} \quad (Y < 0). \tag{33}$$

If we want to prove the existence of ten small-amplitude limit cycles, we need to find the ϵ-order Lyapunov constants ϵV_{1i}, $i = 0, 1, 2, \ldots$. First, we have $V_{10} = 2\pi\delta$, thus letting $\delta = 0$ yields $V_{10} = 0$. A direct computation in higher Lyapunov constants shows that we may set the nonused parameters $q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = 0$, and choose p_4 as a free parameter, and so, without of loss of generality, let $p_4 = 1$. Then, we obtain $V_{11} = \frac{2}{3}(p_1 + p_5)$. Setting $V_{11} = 0$ results in $p_5 = -p_1$, and then V_{12} is simplified to

$$V_{12} = \frac{\pi}{16}[2b_5(p_2 - 1) - 2p_2 + 3p_6 - 2].$$

Letting

$$p_6 = -\frac{2}{3}[b_5(p_2 - 1) - p_2 - 1],$$

we have $V_{12} = 0$. To obtain maximal number of small-amplitude limit cycles bifurcating from the origin of system (33), we first assume that

$$\begin{aligned} F_0 = {} & (2b_5 - 3)(b_2 - b_6)[5b_5(b_2 - 15b_6) - 8b_2 - 69b_6][(10b_2^2 + 30b_2b_6 + 60b_3 + 21)b_5 - 12b_5^3 - (20b_3 - 20)b_5^2 \\ & - 10b_2^2 - 15b_2b_6 + 45b_6^2 - 45b_3 - 36]\{4(b_2 + b_6)b_5^3 + 4[(b_3 - 3)b_2 + 3b_3b_6]b_5^2 - [2b_2^2(b_2 + 6b_6) \\ & + 3(6b_6^2 + 4b_3 - 3)b_2 + 9(4b_3 + 3)b_6]b_5 + 2b_2^3 + 9b_2^2b_6 + 9b_2b_3 - 27b_6(b_6^2 - b_3 - 1)\} \neq 0. \end{aligned}$$

Then, we have

$$V_{13} = -\frac{2}{45}[3p_1(4b_5^2 - 9) + 2b_2(4b_5p_2 + 2b_5 + 1) + 4p_2(3b_6 - 2b_2) + 3p_3(3 - 2b_5)],$$

$$
\begin{aligned}
V_{14} &= \frac{\pi}{96(3-2b_5)}\{(3-2b_5)^2[3p_1(b_2-2b_6) - p_2(3b_5+5b_3+4) - 2b_3 - 3] \\
&\quad + (3-2b_5)[3p_1(b_2+3b_6) + 4b_3b_5 + 15] + 5[2b_2(b_5-1)+3b_6][p_2(b_2+3b_6)+2b_2]\}, \\
V_{15} &= \frac{32}{1575[(10b_2^2+30b_2b_6+60b_3+21)b_5 - 12b_5^3 - (20b_3-20)b_5^2 - 10b_2^2 - 15b_2b_6 + 45b_6^2 - 45b_3 - 36]}F_{15}, \\
V_{16} &= \frac{\pi}{1152(b_2-b_6)}\{[105b_2(b_2-b_6) - b_3(5b_5-8) - b_5(14-5b_5) - 15](b_2-b_6) \\
&\quad + b_6(10b_5+11)(5b_5+7b_3+6)\},
\end{aligned} \tag{34}
$$

where

$$
\begin{aligned}
F_{15} &= -15b_2(3-2b_5)^4 + \frac{1}{2}(3-2b_5)^3[45p_1(b_2^2-b_6^2) + 10b_3(13b_2-3b_6) + 326b_2] \\
&\quad + \frac{1}{2}(3-2b_5)^2[45p_1(b_6-b_2)(b_2+3b_6)(3+2b_3) + 6b_6(60b_2^2+30b_3^2+55b_3+2) \\
&\quad - 2b_2(90b_3^2-20b_2^2+425b_3+448)] + 5(b_2+3b_6)\{9p_1(b_2-b_6)(b_2+3b_6)[2b_2(b_5-1)+3b_6] \\
&\quad - 18b_6^2(1+3b_3) - 6b_2b_6(6b_3b_5-15b_3-10b_5-13) + 4b_2^2(9b_3b_5-9b_3+31b_5-32)\}.
\end{aligned}
$$

Now, we linearly solve the polynomial equations in (34) one by one, i.e. using p_3 to solve for $V_{13}=0$, p_2 for $V_{14}=0$, p_1 for $V_{15}=0$, and b_3 for $V_{16}=0$. Then, higher Lyapunov constants are obtained as follows:

$$
\begin{aligned}
V_{17} &= -\frac{64F_{17}}{33075[5b_5(b_2-15b_6) - 8b_2 - 69b_6]}, \\
V_{18} &= \frac{F_{18}}{20321280[5b_5(b_2-15b_6) - 8b_2 - 69b_6]^2(b_2-b_6)}, \\
V_{19} &= -\frac{F_{19}}{6706022400[5b_5(b_2-15b_6) - 8b_2 - 69b_6]^2(b_2-b_6)}, \\
V_{110} &= \frac{F_{110}}{804722688000[5b_5(b_2-15b_6) - 8b_2 - 69b_6]^3(b_2-b_6)},
\end{aligned}
$$

where

$$
\begin{aligned}
F_{17} &= (b_2-b_6)^2(18200b_2^2 - 8400b_2b_6 + 1610b_5^2 - 2671b_5 - 4112) \\
&\quad - b_6[(b_2-b_6)(980b_5^2+6220b_5+4993) - 28b_6(2b_5+5)(10b_5+11)]
\end{aligned}
$$

and

$$
\begin{aligned}
F_{18} &= 1323\pi\{(b_2-b_6)^3[2100(547+1430b_5)b_2^4 - 12600b_6(3685b_5+4058)b_2^3 + 6(44275b_5^3 + 3465000b_5b_6^2 \\
&\quad + 30445b_5^2 + 4180050b_6^2 - 276886b_5 - 281212)b_2^2 - 3b_6(1382150b_5^3 - 93975b_5^2 - 4960304b_5 \\
&\quad - 3421092)b_2 + 21b_6^2(112200b_5^3 + 657205b_5^2 + 979356b_5 + 428678) + 4(10b_5^2 - 17b_5 - 23) \\
&\quad \times (1190b_5^2 - 1864b_5 - 3385)] - b_6(10b_5+11)[3(b_2-b_6)^2(3920b_5^3 + 200b_5^2 - 21480b_5 - 15676 \\
&\quad - 7b_6^2(5060b_5^2 + 39640b_5 + 28523)) + 28b_6(10b_5+11)(3(b_2-b_6)(154b_5b_6^2 - 12b_5^2 + 385b_6^2 - 26b_5 - 2) \\
&\quad - 2b_6(10b_5+11)(2b_5+1))]\} + [33868800(3-4b_5)](b_2-b_6)(5b_2b_5 - 75b_5b_6 - 8b_2 - 69b_6)^2V_{17},
\end{aligned}
$$

while F_{19}, F_{110} are lengthy polynomials in b_2, b_5, and b_6, which are omitted here. Therefore, the best choice for obtaining maximal number of limit cycles is to find the solutions of b_2, b_5 and b_6 such that $F_{17} = F_{18} = F_{19} = 0$, but $F_0 F_{110} \neq 0$, which results in at most ten small-amplitude limit cycles from the origin of system (33).

To find the solutions of

$$F_{17} = F_{18} = F_{19} = 0,$$

we use the Maple built-in command "rem" to simplify their expressions, yielding

$$\begin{aligned} F_{87} &= \mathrm{rem}(F_{18}, F_{17}, b_5) \\ &= \frac{27\pi}{2(23b_2^2 - 60b_2b_6 + 45b_6^2)^3} F_{87a}, \end{aligned}$$

where F_{87a} is a polynomial in b_2, b_5, and b_6, and linear about b_5, which are omitted here. According to the Remainder Theorem, $F_{17} = F_{18} = F_{19} = 0$ is equivalent to $F_{17} = F_{87a} = F_{19} = 0$. Solving b_5 from $F_{87a} = 0$, we obtain

$$b_5 = -\frac{b_{5N}}{b_{5D}},$$

where

$$\begin{aligned} b_{5N} =\ & 219945320000b_2^{13} - 8896444508000b_2^{12}b_6 + (129149862030000b_6^2 - 87591250880)b_2^{11} \\ & - (868029379700000b_6^3 - 12562848883035b_6)b_2^{10} + (3345344687930000b_6^4 - 20459719846720b_6^2 \\ & - 4738906168)b_2^9 - (8240034473160000b_6^5 - 152899275172525b_6^3 - 520302188888b_6)b_2^8 \\ & + (13705861662108000b_6^6 - 569208248434280b_6^4 - 386091386944b_6^2)b_2^7 - (15773635626840000b_6^7 \\ & - 1232657035887270b_6^5 + 332372718720b_6^3)b_2^6 + (12584085919020000b_6^8 - 1685410122400200b_6^6 \\ & + 13385403764464b_6^4)b_2^5 - (6819306788700000b_6^9 - 1498485253403250b_6^7 - 2940407318736b_6^5)b_2^4 \\ & + (2385010797750000b_6^{10} - 851949290215800b_6^8 - 6880814179200b_6^6)b_2^3 - (481965592500000b_6^{11} \\ & - 286231305920175b_6^9 - 2809982715840b_6^7)b_2^2 + (42195431250000b_6^{12} - 45645686919000b_6^{10} \\ & + 525511658088b_6^8)b_2 + 1231503800625b_6^{11} - 38222898840b_6^9, \end{aligned}$$

$$\begin{aligned} b_{5D} =\ & 1160374359375b_6^{11} - 45292107065625b_2b_6^{10} + (292356046355625b_2^2 - 40040179200)b_6^9 \\ & - (890092355016375b_2^2 - 579318143040)b_2b_6^8 + (1579654918311750b_2^2 + 2573300866176)b_2^2b_6^7 \\ & - (1763064584779050b_2^2 + 6327075037824)b_2^3b_6^6 + (1261988966671650b_2^2 + 1987439043456)b_2^4b_6^5 \\ & - (566375602962750b_2^2 - 2395459177728)b_2^5b_6^4 + (147025015679675b_2^2 - 1009058988160)b_2^6b_6^3 \\ & - (17634219652525b_2^2 + 129869074304)b_2^7b_6^2 + (245426884325b_2^2 - 9098488192)b_2^8b_6 \\ & + (28121213925b_2^2 + 1653808320)b_2^9. \end{aligned}$$

Substituting the solution b_5 into F_{17} and F_{19} yields that

$$\begin{aligned} F_{17} =\ & \frac{392(b_2 - b_6)^3(23b_2^2 - 60b_2b_6 + 45b_6^2)^3}{b_{5D}^2}(175b_2^4 - 5425b_2^3b_6 + 44625b_2^2b_6^2 - 39375b_2b_6^3 \\ & - 41b_2^2 + 1986b_2b_6 - 153b_6^2)^2 F_{17a}, \end{aligned}$$

$$F_{19} = -\frac{4(b_2 - b_6)^3}{b_{5D}^5}(175b_2^4 - 5425b_2^3b_6 + 44625b_2^2b_6^2 - 39375b_2b_6^3 - 41b_2^2 + 1986b_2b_6 - 153b_6^2)^2 F_{19a},$$

where

$$\begin{aligned} F_{17a} =\ & 5332239584000000b_2^{11} - 56945564659200000b_2^{10}b_6 + (270469252818240000b_6^2 + 36789698802325)b_2^9 \\ & - (754278239657600000b_6^3 + 237458529796985b_6)b_2^8 + (1372163893220400000b_6^4 \end{aligned}$$

$$+893695625654020b_6^2+495943470320)b_2^7-(1708735012664400000b_6^5+2231123598503940b_6^3$$
$$-1333793971504b_6)b_2^6+(1484539526156400000b_6^6+3487761622866150b_6^4-3671375629776b_6^2)b_2^5$$
$$-(898075230852240000b_6^7+3295601120874510b_6^5+4948448461200b_6^3)b_2^4+(369686454656400000b_6^8$$
$$+1821076083790740b_6^6+125267865323304b_6^4)b_2^3-(98244966582000000b_6^9+543391572597300b_6^7$$
$$+4836273089904b_6^5)b_2^2+(15100338330000000b_6^{10}+67909069990125b_6^8-894097252080b_6^6)b_2$$
$$-1012690350000000b_6^{11}+342720669375b_6^9+400141101160b_6^7$$

and F_{19a} is a polynomial in b_2 and b_6, which are omitted here. Now, we need to find solutions of b_2 and b_6 such that $F_{17a}=F_{19a}=0$, but $F_0b_{5D}F_{110}\neq 0$. Again, we use the Maple command "rem" and "resultant" to obtain $\text{rem}(F_{19a},F_{17a},b_6)=F_{97}$, and then

$$\begin{aligned}F_{17a97}&=\text{resultant}(F_{17a},F_{97},b_6)\\
&=Cb_2^{296}(536300518636081500000000000b_2^{10}+5264245855401885313684240000b_2^8\\
&\quad+5444511683182115215483371100b_2^6-3877657570922662731034902960b_2^4\\
&\quad-2552578076487729652846663341b_2^2-323454189798917447051504 34)\\
&\quad\times(741986003638863750976562500000000b_2^{16}+104736494519761723876660156250000 0b_2^{14}\\
&\quad+541377423879749549575886962890625b_2^{12}+124762729270469072104017978515625b_2^{10}\\
&\quad+130548059343021917498175864843 75b_2^8+731541816166608967815824681250b_2^6\\
&\quad+169147372235813628224814303 75b_2^4+90788120259696742450828 05b_2^2\\
&\quad+247960811458371923309 7)^8(615601902373365446219763960910156250000b_2^{10}\\
&\quad-731907810934317520907061383859175170625b_2^8\\
&\quad-468737289830045377055886038650808852550b_2^6\\
&\quad-169371759194529046102296126665357207785b_2^4\\
&\quad-353690927917112505903442270661328588 8b_2^2\\
&\quad-478398839038582452695758826159662 08)^{10},\end{aligned}$$

where C is a constant. Finally, we solve $F_{17a97}=0$ to find the solutions of b_2. It can be shown that this polynomial has five real roots, which in turn yield five corresponding solutions for b_6. By checking that $F_{17a}=F_{19a}=0$ and $F_0b_{5D}F_{110}\neq 0$, we found that only two of them satisfy the original equations. We take one of the solutions:

$$b_2=-0.6499316542\ldots,$$
$$b_6=-1.4007939402\ldots.$$

Then, the other perturbation parameters are equal to

$$p_5=-0.6593001227\ldots,$$
$$p_6=-0.3425204361\ldots,$$
$$p_3=-0.7686180478\ldots,$$
$$p_2=0.1486175491\ldots,$$
$$p_1=0.6593001227\ldots,$$
$$b_3=0.6172899175\ldots,$$
$$b_5=-1.9525868799\ldots.$$

The above critical values can be used to define a critical point, called p_c, for which the ϵ-order Lyapunov constants become

$$V_{1i}=0,\quad i=1,2,\ldots,9,$$
$$V_{110}=0.0157763313\cdots\neq 0.$$

Moreover, a direct calculation shows that

$$\det\left[\frac{\partial(V_{17}, V_{18}, V_{19})}{\partial(b_2, b_5, b_6)}\right] = 0.0003020524\cdots \neq 0,$$

implying, by Theorem 2, that system (33) can indeed have ten small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Thus, system (32) can have ten limit cycles around $(1, 0)$ or $(-1, 0)$.

Finally, we check if we can have limit cycles bifurcating from the origin of system (32). A simple direct computation shows that the zero-order Lyapunov constant of system (32) at the origin is equal to

$$V_{00} = \frac{1}{2}(\epsilon\delta - \epsilon)\pi + \frac{1}{2}(\epsilon\delta)\pi = \frac{1}{2}(2\delta - 1)\pi\epsilon \approx -\frac{1}{2}\pi\epsilon \quad \text{for } \delta = o(\epsilon),$$

which implies that the origin of system (32) is a stable focus ($\epsilon > 0$), and so no more small-amplitude limit cycles can bifurcate from the origin of system (32).

If we do not assume $p_7 = p_9 = q_7 = q_9 = 0$, $p_8 = -p_1$, $q_8 = -q_1$, $p_4 = 1$ and q_i, $i = 1, 2, 3, 4, 5, 6$ are nonzero, then the zero-order Lyapunov constant of system (32) at the origin becomes

$$\begin{aligned} V_{00} &= -\frac{1}{2}\epsilon(p_7 - p_{10} - \delta)\pi - \frac{1}{2}\epsilon(q_7 - q_{10} - \delta)\pi \\ &= \frac{1}{2}(2\delta - p_7 - q_7 + p_{10} + q_{10})\pi\epsilon \\ &= \frac{1}{2}(2\delta - p_4 - q_4 - 3p_7 - 3q_7)\pi\epsilon, \end{aligned}$$

where $p_{10} = -p_4 - 2p_7$ and $q_{10} = -q_4 - 2q_7$ have been used. Thus, when $\delta = 0$, we may set $p_4 + q_4 + 3p_7 + 3q_7 = 0$ to get $V_{00} = 0$ for which we may apply proper perturbations to obtain more limit cycles bifurcating from the origin. However, in this case, it can be shown that $p_4 + q_4 + 3p_7 + 3q_7$ is a common factor in the last four Lyapunov constants focus: V_{17}, V_{18}, V_{19} and V_{110}. This indicates that when ten small-amplitude limit cycles exist around the singular point of $(1, 0)$ of system (32), it is not possible to have more limit cycles bifurcating from the origin. In the same way, we know that when ten small-amplitude limit cycles exist around the singular point of $(-1, 0)$ of system (32), it is also not possible to have more limit cycles bifurcating from the origin.

The proof of Theorem 6 is complete. ■

5. Conclusion

In this paper, we have considered a class of planar switching differential systems with cubic homogeneous nonlinearities, and gave a new version of Gasull–Torregrosa method for computing the Lyapunov constants of the planar switching systems. We obtained the center conditions and proved the existence of eight limit cycles for a class of cubic switching systems using this method with the aid of Maple. Moreover, we used one of the center conditions to construct a special integrable system and then perturbed this system to obtain ten small-amplitude limit cycles around the singular point either $(1, 0)$ or $(-1, 0)$, which is a new lower bound on the maximal number of small-amplitude limit cycles obtained around one singular point in such cubic switching systems with cubic homogeneous nonlinearities.

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